

Section 4: The Simplex Method (Ch.4 of text) and includes a further review of linear algebra

A. Aim and Methodology

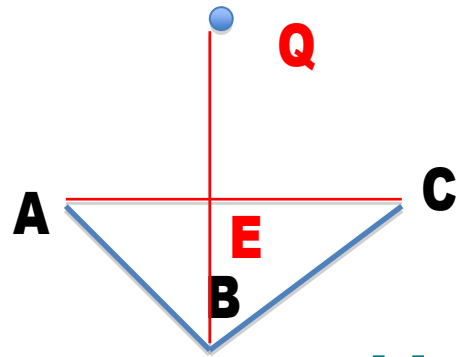
1. proceed from one corner point (extreme point) to another in a manner that the objective function which we want to minimize is decreasing

a. in practice we will proceed along the edges of the boundary of the simplex from one corner point to an adjacent corner point so that the objective function is decreasing.

Def. 4.1: a LOCAL EXTREME POINT OPTIMUM is an extreme point of the feasible region where the value of the objective function is less than or equal to the value of the objective function at all its neighbours, that is, at all the extreme points attached to it by an edge.

Theorem 4.1: If \mathbf{x}_0 is a local extreme point optimum for an LP problem, then \mathbf{x}_0 is a global optimum (that is, it an optimal point) for the LP problem.

We will not give a formal proof but instead look at a picture in the plane, which, by extending it to higher dimensions does provide a formal proof.



Denote the objective function by $f(\mathbf{x})$ and suppose $f(B) \leq f(A)$ and $f(B) \leq f(C)$. Suppose there exists a point Q such that $f(Q) < f(B)$.

Join B and Q . Then, by the convexity of the feasible region, the line BQ is in the feasible region.

Join A and C . Again by convexity the line AC is in the feasible region.

These lines intersect at a point E .

Since E lies on the line AC, then $f(E)$ is an average of the values $f(A)$ and $f(C)$ so that

$$\mathbf{\min(f(A), f(C)) \leq f(E) \leq \max(f(A), f(C)) \quad (1)}$$

But $f(B) \leq f(A)$ and $f(B) \leq f(C)$ so that

$$\mathbf{f(B) \leq \min(f(A), f(B)) \quad (2)}$$

and combining (1) and (2) gives

$$\mathbf{f(B) \leq \min(f(A), f(C)) \leq f(E) \leq \max(f(A), f(C)) \quad (3)}$$

Since E lies on the line BQ, we also get

$$\mathbf{\min(f(B), f(Q)) \leq f(E) \leq \max(f(B), f(Q))}$$

Moreover since E is an interior point of the line, we really have

$$\mathbf{\min(f(B), f(Q)) < f(E) < \max(f(B), f(Q))}$$

But if we are assuming $f(Q) < f(B)$ then

$$\mathbf{f(Q) = \min(f(B), f(Q)) < f(E) < \max(f(B), f(Q)) = f(B) \quad (4)}$$

so from (4) we conclude that

$$\mathbf{f(Q) < f(E) < f(B) \quad (5)}$$

while from (3) we have

$$\mathbf{f(B) \leq f(E) \quad (6)}$$

But (6) contradicts (5). So we cannot assume there exists an extreme point Q such that $f(Q) < f(B)$.

2. local minimum is global minimum

We illustrate the procedure with the Reddy Mikks Co. LP problem in standard form.

$$\min z = -3x_E - 2x_I$$

s.t.

$$x_E + 2x_I + y_1 = 6 \quad (1)$$

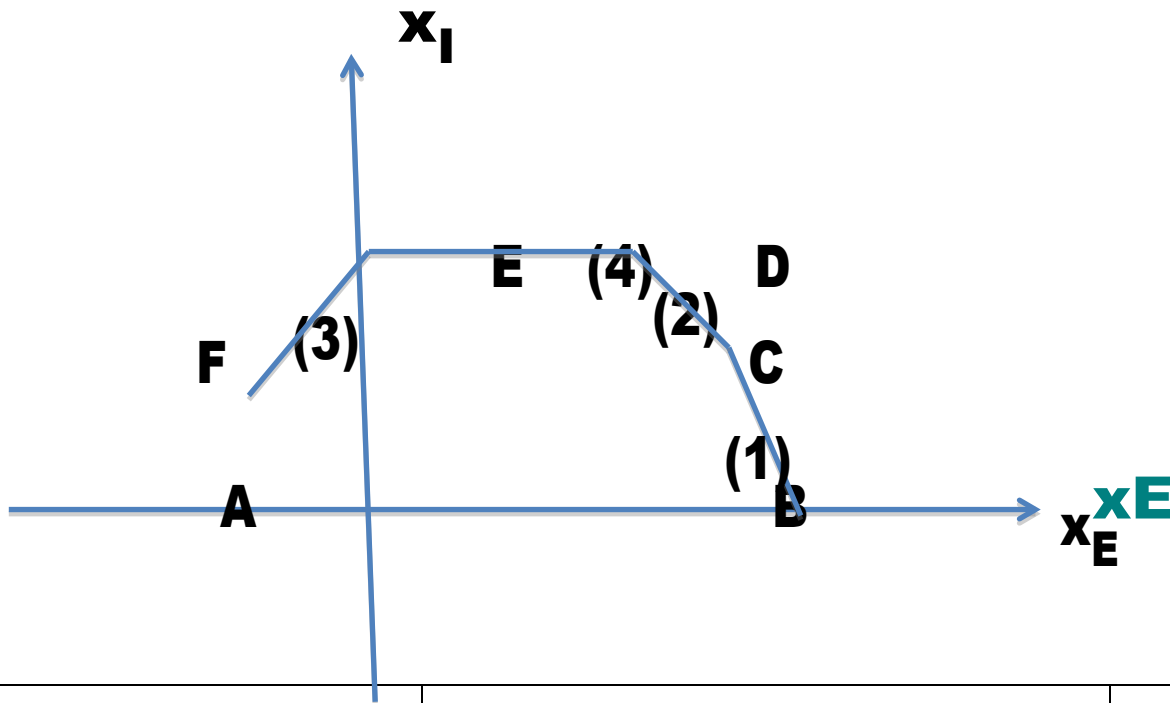
$$2x_E + x_I + y_2 = 8 \quad (2)$$

$$-x_E + x_I + y_3 = 1 \quad (3)$$

$$x_I + y_4 = 2 \quad (4)$$

$$x_E \geq 0, x_I \geq 0, y_i \geq 0 \quad i=1, \dots, 4$$

We look at the graph of the feasible region for x_E, x_I



Extreme Point	Coordinates (x_E, x_I)	Obj. Function z
A	(0,0)	0
B	(4,0)	-12
C	(10/3,4/3)	-38/3
D	(2,2)	-10
E	(1,2)	-5
F	(0,1)	-2

How do we move from point to point?

Extreme Point	Zero Variables (non-basic)	Nonzero variables (Basic Variables)
A	x_E, x_1	y_1, y_2, y_3, y_4
B	x_1, y_2	y_1, x_E, y_3, y_4
C	y_1, y_2	x_1, x_E, y_3, y_4
D	y_1, y_4	x_1, x_E, y_3, y_2
E	y_3, y_4	x_1, x_E, y_1, y_2
F	y_3, x_E	x_1, y_4, y_1, y_2

We note that

(1) At extreme points each must have exactly 2 variables equal to 0.

(2) Adjacent extreme points differ in only one of the zero variables.

(3) (1) is a **unique property of extreme points.**

B. Basic Solutions

Def. 4.2: a **BASIC SOLUTION is a solution resulting from setting $n-m$ of the variables equal to zero where the columns corresponding to the non-zero variables are linearly independent and span \mathcal{R}^m .**

Def. 4.2a: a **BASIC FEASIBLE SOLUTION is a basic solution which is also feasible.**

1. maximum number of basic solutions = $\binom{n}{n-m} = \frac{n!}{m!(n-m)!}$

C. Review of some needed linear algebra

Using definitions 2.3 (linear combination of vectors) and 2.4 (linear independence of vectors) we can state linear independence of vectors in a slightly different way.

Def. 4.3: The vectors $\underline{u}_1, \underline{u}_2, \dots, \underline{u}_k$ are said to be **LINEAR INDEPENDENT** iff $\underline{a}_1 \underline{u}_1 + \underline{a}_2 \underline{u}_2 + \dots + \underline{a}_k \underline{u}_k = \underline{0} \Rightarrow \underline{a}_1 = \underline{a}_2 = \dots = \underline{a}_k = \underline{0}$.

Example 4.1: \mathcal{R}^1 : only a single vector can be lin. indep.

Example 4.2: \mathcal{R}^2 : Any two vectors not in the same or opposite direction are lin. indep.

Example 4.3: In \mathcal{R}^n , any set of vectors containing more than n vectors (e.g. $n+1$) is lin. dep.

Example 4.4: In \mathcal{R}^n , the Euclidean unit coordinate vectors

$\mathbf{e}_1 = (1, 0, 0, \dots, 0), \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, 0, 0, \dots, 1)$ are lin indep.

(Ex. prove this.)

Def. 4.4: A **BASIS** of a vector space V is a set of vectors

$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}, \mathbf{v}_i \in V, \quad i = 1, \dots, m$ such that

(i) $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are lin. indep., and

(ii) $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\} = V.$

Def. 4.4a: A collection of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m\}$ span the space

V iff any vector $\mathbf{v} \in V$ can be written as a l.c. of $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$;

i.e. there exist constants c_1, \dots, c_m such that

$$\mathbf{v} = c_1 \mathbf{v}_1 + \dots + c_m \mathbf{v}_m.$$

Example 4.5: In \mathcal{R}^2 , any two non-parallel vectors from a basis.

Example 4.6: In \mathcal{R}^3 , any three vectors not in the same plane form a basis.

Def. 4.5: an $m \times n$ MATRIX **A** is an array
$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_{11} & \cdots & \mathbf{a}_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{m1} & \cdots & \mathbf{a}_{mn} \end{pmatrix}.$$
 For notation, we sometimes denote $\mathbf{A} = (\mathbf{a}_{ij})$ or $\mathbf{A} = (\mathbf{a}_{ij})_{m,n}$. The columns of **A** can be viewed as vectors in \mathcal{R}^m while the rows of **A** can be viewed as vectors in \mathcal{R}^n .

Def. 4.6: The **COLUMN RANK** of **A**, denoted by $\text{col}(\mathbf{A})$ is the maximum number of lin. indep. columns of **A**.

Def. 4.7: The **ROW RANK** of **A**, denoted by $\text{row}(\mathbf{A})$ is the maximum number of lin. indep. rows of **A**.

Theorem 4.2: col(A)=row(A).

Proof: omitted

Example 4.7: $A = \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 0 \end{pmatrix}$

row(A)=3 (all rows) col(A)=3 (columns 1,2 and 4)

Based on Theorem 4.2, we can make the following definition:

Def. 4.8: The **RANK of a matrix A is defined as the column rank of A (which is equal to the row rank of A).**

Def. 4.9: A matrix A is called a **SQUARE MATRIX if the number of rows equals the number of columns.**

Example 4.8: $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a 2x2 square matrix.

Def. 4.10: An nxn matrix \mathbf{A} is called **INVERTIBLE** iff there exists an nxn matrix denoted by \mathbf{A}^{-1} such that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, where \mathbf{I} is the identity matrix.

Note: \mathbf{I} is the nxn matrix $\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix}$ that is the matrix

which is everywhere 0 off the diagonal and 1 along the diagonal.

Example 4.9: If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $\mathbf{A}^{-1} = \begin{pmatrix} -2 & 1 \\ 3 & -1 \\ 2 & 2 \end{pmatrix}$

CHECK: $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 3 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Note: Show that $\mathbf{AA}^{-1} = \mathbf{I} \Leftrightarrow \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$ Hence to check it is enough to do only one of the multiplications.

D. Solution of Equations

1. n equations in n unknowns

$$\begin{array}{rcll} \mathbf{a}_{11}\mathbf{x}_1 + \dots + \mathbf{a}_{1n}\mathbf{x}_n & = & \mathbf{b}_1 & (1) \\ \vdots & & \vdots & \vdots \\ \mathbf{a}_{n1}\mathbf{x}_1 + \dots + \mathbf{a}_{nn}\mathbf{x}_n & = & \mathbf{b}_n & (n) \end{array}$$

1a. matrix form:

Let $\mathbf{A} = \begin{pmatrix} \mathbf{a}_{11} & \cdots & \mathbf{a}_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n1} & \cdots & \mathbf{a}_{nn} \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{pmatrix}$. Then

$\mathbf{Ax} = \mathbf{b}$ represents this system of equations in matrix form.

Example 4.10: $2x + 3y = 5$ (1)
 $3x + 4y = 6$ (2)

In matrix form

$$\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

How do we solve it?

Method 1: 2 equations in 2 unknowns can be solved very easily using algebraic methods you learned in elementary secondary school. However, we shall use this easy example to illustrate a

general matrix method.

The matrix $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ has an inverse, namely $\begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$.

CHECK: $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} -8+9 & 6-6 \\ -12+12 & 9-8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We multiply both sides of the matrix equation by the matrix

$\begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix}$. (This is an extension of the idea of multiplying

both sides of an *equation* by the same nonzero constant, to *matrix equations*. What we are really doing is taking multiples of each equation and adding them which is permissible. We get

$$\begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

and simplifying yields

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4 & 3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 5 \\ 6 \end{pmatrix} = \begin{pmatrix} -20 + 18 \\ 15 - 12 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \text{ that is}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}. \text{ Now 2 vectors are equal iff each of the}$$

components of the vectors are equal; i.e. $x=-2$ and $y=3$.

CHECK: $2(-2) + 3(3) = -4 + 9 = 5$ (1) **so our answer is correct.**
 $3(-2) + 4(3) = -6 + 12 = 6$ (2)

Question 1: Can a non-square matrix have an inverse?

Answer: We have only defined inverse for square matrix so the straightforward answer is NO.

Question 2: How do we know if an $n \times n$ matrix A has an inverse?

Solution: (1) If $\text{rank}(A)=n$.

(2) If $\det(\mathbf{A}) \equiv |\mathbf{A}| \neq 0$

(3) If \mathbf{A} is 1-1.

(4) If $\mathbf{A}\mathbf{v} = \mathbf{0} \Leftrightarrow \mathbf{v} = \mathbf{0}$

Def. 4.11: A system of equations is called **CONSISTENT if it has at least one solution. Otherwise it is called **INCONSISTENT**.**

Def. 4.12: A system of consistent equations is called **REDUNDANT if the removal of one or more of the set of equations does not change the solution set. In terms of the $m \times n$ matrix \mathbf{A} , representing the equations in matrix form, $\text{rank}(\mathbf{A}) = m \leq n$**

Example 4.11: The following system is redundant:

$$\mathbf{x}_1 + 2\mathbf{x}_2 + 3\mathbf{x}_3 + \mathbf{x}_4 = 5 \quad (1)$$

$$\mathbf{x}_1 + \mathbf{x}_3 - \mathbf{x}_4 = 1 \quad (2)$$

$$3\mathbf{x}_1 + 4\mathbf{x}_2 + 7\mathbf{x}_3 + \mathbf{x}_4 = 11 \quad (3)$$

Note that equation (3) is redundant since it can be obtained from equations (1) and (2) as follows: $2(1)+(2)=(3)$

In vector terms: $\begin{pmatrix} 1 & 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & -1 \end{pmatrix}, \begin{pmatrix} 3 & 4 & 7 & 1 \end{pmatrix}$ are lin. dependent vectors.

Note: Eliminating the redundancy does not affect the solution(s).

Example 4.12: (inconsistency) Consider the inequalities:

$$\mathbf{x}_1 \leq 2 \quad (1)$$

$$\mathbf{x}_1 \geq 4 \quad (2)$$

which in standard form are the equations

$$\mathbf{x}_1 + \mathbf{s}_1 = 2 \quad (1)$$

$$\mathbf{x}_1 - \mathbf{s}_2 = 4 \quad (2)$$

$$\mathbf{x}_1 \geq 0, \mathbf{s}_i \geq 0, i = 1, 2$$

The inconsistency of the equations shows up as $s_2 < 0$.

A. LP Problem in Standard Form

Def. 4.13: An LP is said to be in standard form iff it is of the form

$$\text{minimize } z = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

subject to

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \quad (1)$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \quad (2)$$

\vdots

\vdots

\vdots

\vdots

\vdots

\vdots

\vdots

\vdots

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \quad (m)$$

$$x_i \geq 0, i = 1, \dots, n$$

where

(1) b_1, b_2, \dots, b_m are nonnegative constants

(2) a_{ij} $i=1,\dots,m;j=1,\dots,n$ are constants

(3) c_1, c_2, \dots, c_n are constants

(4) x_1, \dots, x_n are variables

1. properties of standard LP are:

a. constraints are equations with right hand side nonnegative

b. variables are all nonnegative

c. objective function is minimized

We will show that any LP problem can be put into standard form. We could prove this as a theorem but we will not. Instead we shall look at examples where we introduce the techniques for putting an LP into standard form.

We may assume that all redundancies have been eliminated.

The Reddy Mikks Co. example

LP problem

$$\max z = 3x_E + 2x_I$$

s.t.

$$x_E + 2x_I \leq 6 \quad (1)$$

$$2x_E + x_I \leq 8 \quad (2)$$

$$-x_E + x_I \leq 1 \quad (3)$$

$$x_I \leq 2 \quad (4)$$

$$x_E \geq 0 \quad x_I \geq 0$$

We wish to turn it into standard form.

Step 1: Change max to min by

$$\min z = -\left(3x_E + 2x_I\right)$$

$$\min z = -3x_E - 2x_I$$

If we minimize this objective function than we maximize the original objective function.

e.g. if $h(x)$ is a function and the largest value occurs at $x=10$ and $h(10)=50$ then the smallest value of $-h(x)$ is -50 and occurs at $x=10$. Conversely if $-50=h(10)\leq -h(x)$ for all x then

$50=h(10)\geq h(x)$ for all x .

Step 2: Introduce slack variables to turn inequalities into equalities

Def. 4.14: Given a \leq inequality whose right hand side is positive, a **SLACK VARIABLE is a new nonnegative variable y which we introduce into the model to convert the \leq inequality into an equality by adding y from the left hand side ("pull in the slack of the inequality and put it into the slack variable")**

In the Reddy Mikks problem, we will have to introduce 4 different slack variables y_1, y_2, y_3, y_4 , one for each inequality.

Standard form of Reddy Mikks LP:

$$\min z = -3x_E - 2x_I$$

s.t.

$$x_E + 2x_I + y_1 = 6 \quad (1)$$

$$2x_E + x_I + y_2 = 8 \quad (2)$$

$$-x_E + x_I + y_3 = 1 \quad (3)$$

$$x_I + y_4 = 2 \quad (4)$$

$$x_E \geq 0 \quad x_I \geq 0 \quad y_i \geq 0, \quad i = 1, \dots, 4$$

Def. 4.15: Given a \geq inequality whose right hand side is positive, a **SURPLUS VARIABLE** is a new nonnegative variable y which we introduce into the model to convert the \geq inequality into an equality by subtracting y from the left hand side ("pull in the surplus of the inequality and put it into the surplus variable")

Recall: production to meet demand at minimum cost

$$\min z = c_1 x_1 + \dots + c_n x_n$$

s.t.

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \geq \delta_1 \quad (1)$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \geq \delta_2 \quad (2)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \geq \delta_m \quad (m)$$

$$x_i \geq 0 \quad i = 1, \dots, n$$

in standard form becomes

$$\min z = \mathbf{c}_1 \mathbf{x}_1 + \dots + \mathbf{c}_n \mathbf{x}_n$$

s.t.

$$\mathbf{a}_{11} \mathbf{x}_1 + \mathbf{a}_{12} \mathbf{x}_2 + \dots + \mathbf{a}_{1n} \mathbf{x}_n - \mathbf{y}_1 = \delta_1 \quad (1)$$

$$\mathbf{a}_{21} \mathbf{x}_1 + \mathbf{a}_{22} \mathbf{x}_2 + \dots + \mathbf{a}_{2n} \mathbf{x}_n \quad - \mathbf{y}_2 = \delta_2 \quad (2)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$\mathbf{a}_{m1} \mathbf{x}_1 + \mathbf{a}_{m2} \mathbf{x}_2 + \dots + \mathbf{a}_{mn} \mathbf{x}_n \quad \quad \quad - \mathbf{y}_m = \delta_m \quad (m)$$

$$\mathbf{x}_i \geq 0, i = 1, \dots, n; \mathbf{y}_j \geq 0, j = 1, \dots, m$$

Bus scheduling problem:

$$\text{Minimize } z = \mathbf{x}_1 + \mathbf{x}_2 + \dots + \mathbf{x}_6$$

s.t.

$$\mathbf{x}_1 \qquad \qquad \qquad + \mathbf{x}_6 \geq 4 \quad (1)$$

$$\mathbf{x}_1 \quad + \mathbf{x}_2 \qquad \qquad \qquad \geq 8 \quad (2)$$

$$\mathbf{x}_2 \quad + \mathbf{x}_3 \qquad \qquad \qquad \geq 10 \quad (3)$$

$$\mathbf{x}_3 \quad + \mathbf{x}_4 \qquad \qquad \qquad \geq 7 \quad (4)$$

$$\mathbf{x}_4 \quad + \mathbf{x}_5 \qquad \qquad \qquad \geq 12 \quad (5)$$

$$\mathbf{x}_5 \quad + \mathbf{x}_6 \geq 4 \quad (6)$$

$$\mathbf{x}_i \geq 0, i = 1, \dots, 6$$

in standard form becomes

$$\text{Minimize } z = \mathbf{x}_1 + \mathbf{x}_2 + \dots + \mathbf{x}_6$$

s.t.

$$\begin{array}{rcccccccc}
x_1 & & & & & +x_6 & -y_1 & & = 4 & (1) \\
x_1 & +x_2 & & & & & -y_2 & & = 8 & (2) \\
& x_2 & +x_3 & & & & & -y_3 & = 10 & (3) \\
& & x_3 & +x_4 & & & & & -y_4 & = 7 & (4) \\
& & & x_4 & +x_5 & & & & & -y_5 & = 12 & (5) \\
& & & & x_5 & +x_6 & & & & & -y_6 & = 4 & (6)
\end{array}$$

$$x_i \geq 0, i = 1, \dots, 6$$

Note: The # of surplus variables = # constraints with \geq with the RHS ≥ 0 . If the RHS < 0 , then multiply through by -1 (which reverses the inequality to a \leq with the RHS > 0) so that we use a slack variable for these constraints.

Example 4.13: (of slack and surplus variables)

$$\min z = 5x_1 - 3x_2 + 2x_3$$

s.t.

$$x_1 - x_2 \leq 10 \quad (1)$$

$$x_1 + x_3 \leq 7 \quad (2)$$

$$x_1 + x_2 + x_3 = 19 \quad (3)$$

$$x_1 \geq 2 \quad (4)$$

$$x_2 + x_3 \geq 1 \quad (5)$$

$$x_i \geq 0 \quad i = 1, \dots, 3$$

In standard form:

$$\min z = 5x_1 - 3x_2 + 2x_3$$

s.t.

$$\begin{array}{rclcl}
\mathbf{x}_1 & -\mathbf{x}_2 & & +\mathbf{y}_1 & = 10 & \text{(1)} & \text{slack} \\
\mathbf{x}_1 & & +\mathbf{x}_3 & & +\mathbf{y}_2 & = 7 & \text{(2)} & \text{slack} \\
\mathbf{x}_1 & +\mathbf{x}_2 & +\mathbf{x}_3 & & & = 19 & \text{(3)} & \\
\mathbf{x}_1 & & & & & -\mathbf{y}_3 & = 2 & \text{(4)} & \text{surplus} \\
& & \mathbf{x}_2 & +\mathbf{x}_3 & & & -\mathbf{y}_4 & = 1 & \text{(5)} & \text{surplus}
\end{array}$$

$$\mathbf{x}_i \geq 0, i = 1, \dots, 3; \mathbf{y}_j \geq 0, j = 1, \dots, 4$$

Example 4.14: (RHS not all nonnegative)

$$\min z = \mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3$$

s.t.

$$\mathbf{x}_1 \quad \quad \quad -\mathbf{x}_3 = -\mathbf{1} \quad (1)$$

$$\mathbf{x}_1 \quad -\mathbf{x}_2 \quad \quad \geq -\mathbf{1} \quad (2)$$

$$\mathbf{x}_1 \quad +\mathbf{x}_2 \quad +\mathbf{x}_3 \leq \mathbf{14} \quad (3)$$

$$-\mathbf{x}_1 \quad \quad \quad \leq -\mathbf{5} \quad (4)$$

$$\mathbf{x}_i \geq \mathbf{0}, i = 1, 2, 3$$

Step 1: Rewrite constraints so that RHS is nonnegative. Note that inequalities change direction when multiplied by -1.

$$-\mathbf{x}_1 \quad \quad \quad +\mathbf{x}_3 = \mathbf{1} \quad (1) \quad \text{mult (1) by } -\mathbf{1}$$

$$-\mathbf{x}_1 \quad +\mathbf{x}_2 \quad \quad \leq \mathbf{1} \quad (2) \quad \text{mult (2) by } -\mathbf{1}$$

$$\mathbf{x}_1 \quad +\mathbf{x}_2 \quad +\mathbf{x}_3 \leq \mathbf{14} \quad (3) \quad \text{none}$$

$$\mathbf{x}_1 \quad \quad \quad \geq \mathbf{5} \quad (4) \quad \text{mult (4) by } -\mathbf{1}$$

Step 2: Put in slack or surplus variable as required.

$$\min \mathbf{z} = \mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3$$

s.t.

$$\begin{array}{rcccccccl} -\mathbf{x}_1 & & & +\mathbf{x}_3 & & & = & \mathbf{1} & \mathbf{(1)} & \mathbf{none} \\ -\mathbf{x}_1 & +\mathbf{x}_2 & & & -\mathbf{y}_1 & & = & \mathbf{1} & \mathbf{(2)} & \mathbf{surplus} \\ \mathbf{x}_1 & +\mathbf{x}_2 & +\mathbf{x}_3 & & & +\mathbf{y}_2 & = & \mathbf{14} & \mathbf{(3)} & \mathbf{slack} \\ \mathbf{x}_1 & & & & & & -\mathbf{y}_3 & = & \mathbf{5} & \mathbf{(4)} & \mathbf{surplus} \end{array}$$

$$\mathbf{x}_i \geq \mathbf{0}, \mathbf{y}_i \geq \mathbf{0} \quad \mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3};$$

B. Free Variable

Def. 4.16: A variable is called a **FREE VARIABLE** if it is unconstrained in sign.

Example 4.15: free variable

$$\min z = \mathbf{x}_1 + \mathbf{2x}_2 - \frac{\mathbf{1}}{\mathbf{2}}\mathbf{x}_3$$

s.t.

$$\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 = 17 \quad (1)$$

$$\mathbf{x}_1 \leq 25 \quad (2)$$

$$\mathbf{x}_2 \leq 19 \quad (3)$$

$$\mathbf{x}_1 + \mathbf{x}_3 \geq 5 \quad (4)$$

$\mathbf{x}_i \geq 0, i = 1, 2$ \mathbf{x}_3 unconstrained in sign

How do we deal with free variables in order to convert the LP into standard form?

Method 1: Let $\mathbf{x}_3 = \mathbf{u}_1 - \mathbf{v}_1$ where $\mathbf{u}_1 \geq 0$ and $\mathbf{v}_1 \geq 0$.

Note: \mathbf{u}_1 and \mathbf{v}_1 are not unique; e.g. if $\mathbf{x}_3 = 5$, we take

$\mathbf{u}_1 = 12$ and $\mathbf{v}_1 = 7$ or $\mathbf{u}_1 = 17$ and $\mathbf{v}_1 = 12$. If

$\mathbf{x}_3 = \mathbf{u}_1^* - \mathbf{v}_1^*$ then replacing \mathbf{u}_1^* and \mathbf{v}_1^* by $\mathbf{u}_1^* + \mathbf{k}$ and $\mathbf{v}_1^* + \mathbf{k}$ will also

work for any \mathbf{k} such that $\mathbf{u}_1^* + \mathbf{k} \geq 0$ and $\mathbf{v}_1^* + \mathbf{k} \geq 0$. That at least

one such pair exists is easy:

if $x_3 \geq 0$, take $u_1 = x_3$ and $v_1 = 0$ since $x_3 = u_1 - 0 = u_1 - v_1$ and

$u_1 = x_3 \geq 0$ and $v_1 = 0 \geq 0$

if $x_3 < 0$, take $u_1 = 0$ and $v_1 = -x_3$ since $x_3 = 0 - (-x_3) = u_1 - v_1$ and

$u_1 = 0 \geq 0$ and $v_1 = -x_3 > 0$ since $x_3 < 0$.

In standard form:

$$\min z = x_1 - 2x_2 - \frac{1}{2}u_1 + \frac{1}{2}v_1$$

s.t.

$$x_1 - x_2 + u_1 - v_1 = 17 \quad (1)$$

$$x_1 + y_1 = 25 \quad (2)$$

$$x_2 + y_2 = 19 \quad (3)$$

$$x_1 + u_1 - v_1 - y_3 = 5 \quad (4)$$

$$\mathbf{x}_i \geq 0, i = 1, 2; \mathbf{y}_j \geq 0, j = 1, 2, 3; \mathbf{u}_1 \geq 0, \mathbf{v}_1 \geq 0$$

Method 2: For the moment, ignore the free variable issue and write the LP in standard form

$$\min z = \mathbf{x}_1 + 2\mathbf{x}_2 - \frac{1}{2}\mathbf{x}_3$$

s.t.

$$\mathbf{x}_1 - \mathbf{x}_2 + \mathbf{x}_3 = 17 \quad (1)$$

$$\mathbf{x}_1 + \mathbf{y}_1 = 25 \quad (2)$$

$$\mathbf{x}_2 + \mathbf{y}_2 = 19 \quad (3)$$

$$\mathbf{x}_1 + \mathbf{x}_3 - \mathbf{y}_3 = 5 \quad (4)$$

$$\mathbf{x}_i \geq 0, i = 1, 2; \mathbf{y}_j \geq 0, j = 1, 2, 3; \mathbf{x}_3 \text{ unrestricted in sign}$$

Solve one of the equations for \mathbf{x}_3 in terms of the other

variables (which are all nonnegative). In this case, say equation (4)

$$\mathbf{x}_3 = 5 - \mathbf{x}_1 + \mathbf{y}_3$$

and substitute wherever \mathbf{x}_3 appears in the other constraints.

Note that equation (4) disappears as a constraint since we are using substitution. ((4) vcomes the tautology 5=5.)

So in our example we get

$$\mathbf{min} \mathbf{z} = \mathbf{x}_1 + 2\mathbf{x}_2 - \frac{1}{2}(5 - \mathbf{x}_1 + \mathbf{y}_3)$$

s.t.

$$\mathbf{x}_1 - \mathbf{x}_2 + 5 - \mathbf{x}_1 + \mathbf{y}_3 = 17 \quad (1)$$

$$\mathbf{x}_1 + \mathbf{y}_1 = 25 \quad (2)$$

$$\mathbf{x}_2 + \mathbf{y}_2 = 19 \quad (3)$$

$$\mathbf{x}_i \geq 0, i = 1, 2; \mathbf{y}_j \geq 0, j = 1, 2, 3;$$

which simplifies to

$$\min z = \frac{3}{2}\mathbf{x}_1 + 2\mathbf{x}_2 - \frac{5}{2} + \frac{1}{2}\mathbf{y}_3$$

s.t.

$$-\mathbf{x}_2 + \mathbf{y}_3 = 12 \quad (1)$$

$$\mathbf{x}_1 + \mathbf{y}_1 = 25 \quad (2)$$

$$\mathbf{x}_2 + \mathbf{y}_2 = 19 \quad (3)$$

$$\mathbf{x}_i \geq 0, i = 1, 2; \mathbf{y}_j \geq 0, j = 1, 2, 3;$$

Note that the objective function is not linear (a problem) because of the $-5/2$. However, this function reaches its optimal value at the same point that the objective function

$z = \frac{3}{2}\mathbf{x}_1 + 2\mathbf{x}_2 + \frac{1}{2}\mathbf{y}_3$ reaches its optimal value so we replace

the nonlinear objective function by the linear objective function to get the problem in standard form.

$$\min z = \frac{3}{2}x_1 + 2x_2 + \frac{1}{2}y_3$$

s.t.

$$-x_2 + y_3 = 12 \quad (1)$$

$$x_1 + y_1 = 25 \quad (2)$$

$$x_2 + y_2 = 19 \quad (3)$$

$$x_i \geq 0, i = 1, 2; y_j \geq 0, j = 1, 2, 3;$$

What are the advantages and disadvantages of the two methods?

(1) Method (1) is simpler to apply (simple replacement) while Method (2) involves solving equations (set of equations if there is more than one free variable) and substitution.

(2) Method (2) has less variables and fewer constraints (a big advantage which sometimes makes up for the extra work involved in solving for the free variable.

(3) Method (1) can more easily be programmed; but method (2) will solve faster once implemented.

C. Matrix Form of Standard LP

1. $\mathbf{A} = (a_{ij})$ $i=1, \dots, m; j=1, \dots, n$ is an $m \times n$ matrix of constants

2. $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ is a $n \times 1$ matrix (i.e. a column vector of length n)

of variables.

3. $\mathbf{c} = (c_1, c_2, \dots, c_n)$ is a $1 \times n$ matrix (i.e. a row vector of length n)
of constants

4. $\underline{\mathbf{b}} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \vdots \\ \mathbf{b}_m \end{pmatrix}$ is an $m \times 1$ matrix (a column vector of length m) of

nonnegative constants; i.e. $\underline{\mathbf{b}} \geq \mathbf{0}$.

Then the LP problem in standard form in matrix notation is

$$\min \mathbf{z} = \underline{\mathbf{c}} \cdot \mathbf{x}$$

s.t.

$$\mathbf{Ax} = \underline{\mathbf{b}}$$

$$\mathbf{x} \geq \mathbf{0}$$

with the above setup.

Again, we assume that all redundancies have been eliminated.

This implies that $m \leq n$.

If $\mathbf{A} = \begin{pmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \dots & \mathbf{a}^n \\ a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ then we have named the columns

by $\mathbf{a}^1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}$, $\mathbf{a}^2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}$, ..., $\mathbf{a}^n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$ and in terms of the

named column vectors we may write

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \dots & \mathbf{a}^n \end{pmatrix}.$$

Since the equations are not redundant, we have $\text{rank}(\mathbf{A}) = m \leq n$. Hence $\text{col}(\mathbf{A}) = m$; i.e. there are m lin indep. columns.

Note: It is possible to have more than one set of m lin. indep. columns; e.g.

Let $A = \begin{pmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 & \mathbf{a}^4 & \mathbf{a}^5 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}$. Then the set of vectors

$\left\{ \mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3 \right\}$ are lin. independent,

$\left\{ \mathbf{a}^2, \mathbf{a}^3, \mathbf{a}^4 \right\}$ are lin. independent,

$\left\{ \mathbf{a}^1, \mathbf{a}^5, \mathbf{a}^2 \right\}$ are lin. independent

etc.

We are now in a position to understand basic and non-basic solutions of the system $\mathbf{Ax} = \mathbf{b}$

For any collection of m lin. indep. columns of \mathbf{A} , they form a basis of \mathcal{R}^m .

Suppose, without loss of generality (wolog) the first m columns of \mathbf{A} are lin. indep.

We will show how to find a basic solution with respect to the basic variables, namely the variables corresponding to the lin. indep. columns which in this case are x_1, x_2, \dots, x_m . We write \mathbf{A}

as $\mathbf{A} = \left(\begin{array}{c|ccc} \mathbf{a}^1 & \dots & \mathbf{a}^m & \mathbf{a}^{m+1} & \dots & \mathbf{a}^n \end{array} \right)$. We will now formally partition \mathbf{A} , namely,

$\mathbf{A} = \left(\begin{array}{c|c} \mathbf{B} & \mathbf{C} \end{array} \right)$ where $\mathbf{B} = \left(\begin{array}{ccc} \mathbf{a}^1 & \dots & \mathbf{a}^m \end{array} \right)$ and $\mathbf{C} = \left(\begin{array}{ccc} \mathbf{a}^{m+1} & \dots & \mathbf{a}^n \end{array} \right)$.

Note that \mathbf{B} is an $m \times m$ (square) matrix while \mathbf{C} is an $m \times (n-m)$ matrix. It is also important to note that $\text{rank}(\mathbf{B})=m$ so that \mathbf{B} is

invertible. Our matrix equation

$$\mathbf{Ax} = \mathbf{b}$$

becomes

$$\left(\mathbf{B} \mid \mathbf{C} \right) \begin{pmatrix} \mathbf{x}_B \\ \mathbf{x}_C \end{pmatrix} = \mathbf{b}$$

which is equivalent to

$$\mathbf{Bx}_B + \mathbf{Cx}_C = \mathbf{b}$$

We set $\mathbf{x}_C = \mathbf{0}$ where $\mathbf{0}$ is the zero vector of length $(n-m)$.

Hence we get

$$\mathbf{Bx}_B = \mathbf{b}$$

so that

$$\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$$

Hence our basic solution is $\left(\mathbf{B}^{-1}\mathbf{b} \mid \mathbf{0} \right)$.

D. Solving Systems of Equations by Gauss-Jordan Row Reduction

1. Gaussian row reduction (Gauss-Jordan+interchange of rows).

Example 4.16: Solve the system of equations

$$x_1 + 5x_3 + 6x_4 = 5 \quad (1)$$

$$5x_3 + 4x_4 = 2 \quad (2)$$

$$-2x_1 + 3x_2 + x_3 + x_4 = 0 \quad (3)$$

$$3x_1 - 7x_3 + x_4 = 3 \quad (4)$$

Note: If the equations are independent, then there is a unique solution since we would have 4 independent equations in 4 unknowns.

Matrix Form:

$$\begin{pmatrix} 1 & 0 & 5 & 6 \\ 0 & 0 & 5 & 4 \\ -2 & 3 & 1 & 1 \\ 3 & 0 & -7 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 0 \\ 3 \end{pmatrix}$$

Adjoin b column to the A matrix:

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & 6 & 5 \\ 0 & 0 & 5 & 4 & 2 \\ -2 & 3 & 1 & 1 & 0 \\ 3 & 0 & -7 & 1 & 3 \end{array} \right) \xrightarrow{\substack{(3)+2(1) \\ (4)-3(1)}} \left(\begin{array}{cccc|c} 1 & 0 & 5 & 6 & 5 \\ 0 & 0 & 5 & 4 & 2 \\ 0 & 3 & 11 & 13 & 10 \\ 0 & 0 & -22 & -17 & -12 \end{array} \right)$$

$$\xrightarrow{\substack{(3) \div 3 \\ (3) \leftrightarrow (2)}} \left(\begin{array}{cccc|c} 1 & 0 & 5 & 6 & 5 \\ 0 & 1 & \frac{11}{3} & \frac{13}{3} & \frac{10}{3} \\ 0 & 0 & 5 & 4 & 2 \\ 0 & 0 & -22 & -17 & -12 \end{array} \right)$$

$$\begin{array}{l} \xrightarrow{(3) \div 5} \\ (1) - 5(3) \\ (2) - \frac{11}{3}(3) \\ (4) + 22(3) \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 7/5 & 28/15 \\ 0 & 0 & 1 & 4/5 & 2/5 \\ 0 & 0 & 0 & 3/5 & -16/5 \end{array} \right)$$

$$\begin{array}{l} \xrightarrow{(4) \div \frac{3}{5}} \\ (1) - 2(4) \\ (2) - \frac{7}{5}(4) \\ (3) - \frac{4}{5}(4) \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 41/3 \\ 0 & 1 & 0 & 0 & 28/3 \\ 0 & 0 & 1 & 0 & 14/3 \\ 0 & 0 & 0 & 1 & -16/3 \end{array} \right)$$

Hence the solution is $\mathbf{x}_1 = \frac{41}{3}$ $\mathbf{x}_2 = \frac{28}{3}$ $\mathbf{x}_3 = \frac{14}{3}$ $\mathbf{x}_4 = -\frac{16}{3}$.

Note that the Gauss-Jordan process shows that the 4

equations are in fact independent and that there is a unique solution (as expected in that case).